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Thermodynamics of the one-dimensional multicomponent Fermi gas with a δ-function interaction

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Abstract. We consider a gas of fermions with parabolic dispersion and N spin-components (or spin S, $N = 2S + 1$) with SU(N) symmetry in one dimension interacting via a δ-function potential. The model is integrable and its solution has been obtained by Sutherland in terms of N nested Bethe *ansatze*. We analyse the discrete Bethe *ansatz* equations and classify their solutions according to the string hypothesis. The thermodynamic Bethe-*ansatz* equations are derived for arbitrary band-filling for both repulsive and attractive interaction in terms of the thermodynamic energy potentials for the classes of eigenstates of the Hamiltonian. These equations are then discussed in several limiting cases, e.g. $S = 1/2$, the ground state ($T \to 0$), for vanishing interaction strength, for strong coupling, in the high-temperature limit, and the large-N limit.

1. Introduction

Highly correlated electron systems were a subject of great interest even before the discovery of high-temperature superconductivity. The strong correlations induce quantum fluctuations, which determine the physical properties of the system. Quantum fluctuations play a particularly important role in systems of reduced dimensionality. Exact solutions are often accessible for one-dimensional models and can serve as a testing ground for approximations intended for more complex problems. It is therefore important to reach an understanding that is as complete as possible for these exactly soluble models.

In this paper we study the equilibrium thermodynamics of a gas of fermions in one dimension with N spin-components (colours with SU(N) symmetry) interacting via a δ-function potential. The fermions are assumed to have a parabolic dispersion with the mass of the particles equated to 1/2 and the interaction strength is denoted by c. The Hamiltonian has then the following form:

$$H = -\sum_{i=1}^{N_e}\left(\frac{\partial}{\partial x_i}\right)^2 + 2c\sum_{i<j}\delta(x_i - x_j) \tag{1.1}$$

where N_e is the number of particles in the system and the sum in the interaction term is restricted to $i < j$ to avoid double counting. The Hamiltonian is independent of the colours of the particles, which are incorporated via the symmetry of the wavefunction.

Model (1.1) has been exactly diagonalized by means of the Bethe *ansatz* for the following situations. (i) If $N = N_e$ the Pauli principle is irrelevant and each particle interacts with all the others, so that the system is equivalent to the gas of bosons solved by Lieb and Liniger [1]. (ii) For $S = 1/2$ the model was diagonalized by two nested Bethe *ansatze* by Gaudin [2] and Yang [3]. (iii) This result was extended by Sutherland

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[4] to an arbitrary number N of spin (colour) components by means of a sequence of N nested Bethe *ansatze*, each step eliminating one spin degree of freedom. The ground-state properties of the Bose gas with repulsive interaction have been discussed in [1,5] and the equations determining the thermodynamics were derived in [6]. For the two-component fermion gas with a repulsive interaction some ground-state properties were obtained in [2,3], while the case of an attractive coupling was briefly discussed in [2]. The thermodynamics for $S = 1/2$ was treated by Takahashi [7] and Lai [8] for both attractive and repulsive interactions. Finally, the ground-state Bethe *ansatz* equations for an arbitrary number of components and $c > 0$ were obtained in [4] (see also [9]), while the ground-state solution for attractive potential can be found in [10] (see also [9]). In this paper we extend the thermodynamic Bethe *ansatz* equations for $S = 1/2$ to an arbitrary number of colours.

Our starting point are Sutherland's discrete Bethe *ansatz* equations [4] for the model (1.1). The coordinate Bethe *ansatz* is formulated as a linear combination of plane waves with N_e wavenumbers given by the set $\{k_j\}$. All the wavenumbers k_j have to be different for the plane waves to be linearly independent. The energy and momentum of the state are given by

$$E = \sum_{j=1}^{N_e} k_j^2 \qquad P = \sum_{j=1}^{N_e} k_j. \tag{1.2}$$

Periodic boundary conditions give rise to a new eigenvalue problem of N_e operators, each consisting of products of $N_e - 1$ two-particle scattering matrices. All N_e operators are to be diagonalized simultaneously. For $S = 1/2$ assume $N_e - M$ electrons with spin up and M with spin down. The problem can be parametrized as an one-dimensional lattice gas of N_e sites with M particles and $N_e - M$ holes ($M \leqslant N_e - M$), or equivalently, as a Heisenberg chain, and can be solved by a second generalized (nested) Bethe *ansatz* [2,3] in terms of another set of rapidities which we denote $\{\xi_\alpha\}$.

The generalization of the Yang–Gaudin solution to more than two colours [4] consists of an iterative application of the Bethe–Yang hypothesis (generalized Bethe *ansatz*), such that one colour is eliminated at each step, leading to N nested Bethe *ansatze* (for N colours). Each Bethe *ansatz* gives rise to a new set of rapidities, $\{\xi_\alpha^{(l)}\}$, $l = 0, \ldots, N-1$, with $k_\alpha = \xi_\alpha^{(0)}$ and α being the running index within each set. All rapidities within a given set have to be different to ensure linearly independent solutions. Consider fermions of spin S, so that $N = 2S + 1$. Let us now denote by N_{S-m} the number of particles with spin component m and define

$$M^{(i)} = \sum_{m=-S+i}^{S} N_{S+m} \qquad M^{(0)} = N_e \qquad M^{(2S+1)} = 0 \tag{1.3}$$

such that $N_e \geqslant M^{(1)} \geqslant \ldots \geqslant M^{(2S)} \geqslant 0$ (note that the Bethe *ansatz* eigenfunctions are only a basis of states within this subspace, but they are not a complete set of eigenstates of the Hamiltonian [11]). Then the nested Bethe *ansatze* yield the following sets of coupled equations (SU(N) symmetry) [4]:

$$\begin{gathered}
\exp(\mathrm{i}\xi_\alpha^{(0)} L) = \prod_{\beta=1}^{M^{(1)}} e(\xi_\alpha^{(0)} - \xi_\beta^{(1)}) \qquad \alpha = 1, \ldots, N_e \\
\prod_{\beta=1}^{M^{(l-1)}} e(\xi_\alpha^{(l)} - \xi_\beta^{(l-1)}) \prod_{\beta=1}^{M^{(l+1)}} e(\xi_\alpha^{(l)} - \xi_\beta^{(l+1)}) = -\prod_{\beta=1}^{M^{(l)}} e\big[(\xi_\alpha^{(l)} - \xi_\beta^{(l)})/2\big] \\
\alpha = 1, \ldots, M^{(l)} \qquad l = 1, \ldots, 2S
\end{gathered} \tag{1.4}$$

where L is the length of the box and

$$e(x) = (\mathrm{i}x - c/2) \,/\, (\mathrm{i}x + c/2). \tag{1.5}$$

The rapidities $\xi_\alpha^{(l)}$ may take real and complex values, and are the solutions of the set of non-linear algebraic equations (1.4). There are many solutions, which are discussed below, each one corresponding to a possible state of the system.

The partition function or free energy of a system can be obtained with the knowledge of all the energy eigenvalues. In the thermodynamic limit there is an infinite number of states, all solutions of (1.4), such that it is necessary to classify them and introduce a density of states for each class of excitations. The structure of the ground and excited states for a *repulsive* interaction is discussed in section 2 and relations among the densities of these states are derived. Sets of thermodynamic equations are obtained by minimizing the free energy functional. Special limits of these equations are considered in section 3, i.e. we recover the ground state in the limit $T \to 0$, discuss the $c \to 0$ (free fermions) and the $c \to \infty$ limits, the high-temperature limit, and the large-N limit. In section 4 we classify the states for an *attractive* interaction and derive the corresponding integral equations satisfied by the thermodynamic energy potentials. The special limits for the attractive case are discussed in section 5. Conclusions and a discussion of some properties of the model follow in section 6. The results of this paper extend the thermodynamic equations derived by Takahashi [7] and Lai [8] for $S = 1/2$ to an arbitrary number of spin components with SU(N) symmetry.

2. Classification of states and thermodynamic equations: repulsive interaction

The partition function of a system can be obtained from a knowledge of all the energy eigenvalues. It is therefore necessary to classify all the solutions of the Bethe *ansatz* equations (1.4) and introduce densities of states for each class of excitations. The equilibrium free energy is then obtained by minimization of the free energy functional with respect to the densities. The mathematical approach to derive the thermodynamic Bethe *ansatz* equations was developed by Yang and Yang [6] for the one-dimensional boson gas with a δ-function interaction and has been extended to numerous other models [7, 8, 12–24]. Models with a classification of states similar to that of the gas of fermions with a *repulsive* interaction are, for instance, the Heisenberg chain [12], the one-dimensional Hubbard model with $U > 0$ [13], the Kondo problem [15, 16] and the Coqblin–Schrieffer model [17–19]. While on one hand our results are the extension of [7, 8] to arbitrary spin, on the other hand, the integral equations have a structure related to that of the Bethe *ansatz* equations for the Coqblin–Schrieffer model of spin S [17–19] and the generalized SU(N) symmetric Heisenberg chain [25].

Below we first classify the solutions of the discrete Bethe *ansatz* equations for a *repulsive* interaction in the thermodynamic limit according to the string hypothesis† and then we derive the thermodynamic Bethe *ansatz* integral equations.

2.1. Classification of states for arbitrary spin

For a *repulsive* interaction, $c > 0$, the charge rapidities k_α are all real, since (Cooper-pair-like) bound states between electrons cannot form [2-4,9]. On the other hand, the spin-rapidities can form strings of arbitrary length $(n-1)$, $n = 1, \ldots, \infty$, and real centre-of-mass

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† The accuracy of the string hypothesis has been questioned by Faddeev and Takhtajan in [26]. Their findings, however, affect neither the counting of states nor the thermodynamic properties.

rapidity $\Lambda_n^{(l)}$

$$\xi_n^{(l)\mu} = \Lambda_n^{(l)} + \mathrm{i}\mu c/2 + \delta^{(l)\mu} \qquad \mu = -(n-1), -(n-3), \ldots, (n-1) \tag{2.1}$$

where $l = 1, \ldots, 2S$ and $\delta_\mu^{(l)} = \mathrm{O}(\mathrm{e}^{-\kappa L})$, $\kappa > 0$, vanishes in the thermodynamic limit, $L \to \infty$. Complex spin rapidities correspond to many-electron spin-states, often referred to as bound states, although they correspond to excited states (however, the wavefunction falls off exponentially as a function of the relative distance between the particles). Below we adopt the traditional denomination of 'spin bound states'. Since there are $2S$ possible spin-flips per electron there are $2S$ classes of spin rapidities and $2S$ sets of strings. If $M_n^{(l)}$ is the number of spin bound states of n electrons of the lth set of rapidities we have that

$$M^{(l)} = \sum_n n M_n^{(l)}. \tag{2.2}$$

Note that the $S = 1/2$ situation [7, 8] is contained as a special case.

The above string solutions are inserted into the discrete Bethe *ansatz* equations (1.4), and after some algebra coupled equations for the real parameters $\{k_\alpha\}$ and $\{\Lambda_{n\alpha}^{(l)}\}$ are obtained. Since these equations represent relations among phase shifts, it is convenient to make them logarithmic. The logarithmized equations are defined modulo 2π. This defines a set of integers (or half integers) $\{I_{n\alpha}^{(l)}\}$ for each class of states ($l = 0$ corresponds to the set of charge rapidities). Since the Bethe *ansatz* requires all rapidities to be different, also all integers I_α within one set must be distinct. These integers play the role of quantum numbers of the many-body system. Each state is characterized by a different collection of quantum numbers. Hence, an arbitrary integer (half integer) is either contained in a given set or it is missing. Consequently, within the Bethe *ansatz* formulation all the rapidities obey Fermi–Dirac statistics. A quantum number that is present is called a 'particle' and one that is missing a 'hole'.

In the thermodynamic limit, i.e. $L \to \infty$ with $M_n^{(l)} \to \infty$ by keeping the ratios constant, the above non-linear equations for the rapidities can be transformed into linear integral equations. In the limit of large L the variable $2\pi I_{n\alpha}^{(l)}/L$ becomes closely spaced and can be regarded as continuous. It is convenient to introduce 'particle' density functions for each set of rapidities: $\rho(k)$ for the charge rapidities and $\sigma_n^{(l)}(\Lambda)$ for the strings (spin bound states) $\Lambda_n^{(l)}$, and similarly the corresponding 'hole' distribution functions. Since any quantum number corresponds to either a particle or a hole, the derivative of $I^{(0)}/L$ with respect to k and of $I_n^{(l)}/L$ with respect to $\Lambda_n^{(l)}$ represents the sum of the particle and hole density of states, i.e. $\rho(k) + \rho_{\mathrm{h}}(k)$ and $\sigma_n^{(l)}(\Lambda) + \sigma_{n\mathrm{h}}^{(l)}(\Lambda)$, respectively. The 'particle' density and the 'hole' density are complementary functions.

As a result, we obtain linearly coupled integral equations for these distribution functions; Fourier transforming we have after some algebra

$$\begin{aligned} \delta(\omega) &= \hat{\rho}(\omega) + \hat{\rho}_{\mathrm{h}}(\omega) - \sum_{n=1}^{\infty} \exp(-n|\omega|c/2)\hat{\sigma}_n^{(1)}(\omega) \\ \hat{\sigma}_{n\mathrm{h}}^{(l)}(\omega) + \sum_{m=1}^{\infty} \hat{A}_{nm}(\omega)\hat{\sigma}_m^{(l)}(\omega) &= \sum_{m=1}^{\infty} [\hat{A}_{nm}(\omega)/2\cosh(\omega c/2)]\big[\hat{\sigma}_m^{(l+1)}(\omega) + \hat{\sigma}_m^{(l-1)}(\omega)\big] \end{aligned} \tag{2.3}$$

where $n = 1, \ldots, \infty$, $l = 1, \ldots, 2S$ and the hat denotes a Fourier transform. We identify $\hat{\sigma}_m^{(0)}(\omega) = \delta_{m,1}\hat{\rho}(\omega)$, $\hat{\sigma}_m^{(N)}(\omega) \equiv 0$ and

$$\hat{A}_{nm}(\omega) = \coth(|\omega c|/2)\{\exp(-|n-m||\omega c|/2) - \exp[-(n+m)|\omega c|/2]\}. \tag{2.4}$$

The above relations among the particle and hole densities of states follow from the solutions of the Bethe *ansatz* equations (string solutions and Fermi statistics) in the thermodynamic limit and are valid under general conditions (thermal equilibrium and non-equilibrium).

The energy, momentum and total number of electrons of the system are given by

$$E/L = \int \mathrm{d}k k^2 \rho(k) \qquad P/L = \int \mathrm{d}k k \rho(k) \qquad N_{\mathrm{e}}/L = \int \mathrm{d}k \rho(k). \tag{2.5a}$$

The number of particles of each colour (i.e. also the magnetization) can be determined through ($l = 1, \ldots, 2S$)

$$M^{(l)} = \sum_{n=1}^{\infty} n \int \mathrm{d}\Lambda \sigma_n^{(l)}(\Lambda). \tag{2.5b}$$

An alternative set of equations that is equivalent to the second set of (2.3) is

$$\begin{aligned} 2\cosh(\omega c/2)\hat{\sigma}_{m\mathrm{h}}^{(l)}(\omega) - \hat{\sigma}_{m+1\mathrm{h}}^{(l)}(\omega) - \hat{\sigma}_{m-1\mathrm{h}}^{(l)}(\omega) & \\ = \hat{\sigma}_m^{(l+1)}(\omega) + \hat{\sigma}_m^{(l-1)}(\omega) - 2\cosh(\omega c/2)\hat{\sigma}_m^{(l)}(\omega) & \qquad m \geqslant 2 \\ 2\cosh(\omega c/2)\hat{\sigma}_{1\mathrm{h}}^{(l)}(\omega) - \hat{\sigma}_{2\mathrm{h}}^{(l)}(\omega) = \hat{\sigma}_1^{(l+1)}(\omega) + \hat{\sigma}_1^{(l-1)}(\omega) - 2\cosh(\omega c/2)\hat{\sigma}_1^{(l)}(\omega) & \end{aligned} \tag{2.6}$$

where again $\hat{\sigma}_m^{(0)}(\omega) = \delta_{m,1}\hat{\rho}(\omega)$ and $\hat{\sigma}_m^{(N)}(\omega) \equiv 0$.

The above relations are similar (except for the driving terms) to those of the Coqblin–Schrieffer impurity model [17–19] and the SU(N) Heisenberg model [25]. Neither of these models, however, has charge fluctuations, present in the electron gas.

2.2. *Thermodynamics for arbitrary spin S*

To obtain the free energy we have to impose thermal equilibrium. For this purpose we consider a free energy functional, $F = E - TS$, of the distribution functions for particles and holes, where E is the energy (given by (2.5a)), T is the temperature and S is the distribution entropy of 'particles' and 'holes'. There is an entropy term for each class of excitations; since particles and holes are governed by Fermi statistics, the entropy term for the charge rapidities, for example, is given by

$$S_\rho/L = \int \mathrm{d}k[(\rho + \rho_{\mathrm{h}})\ln(\rho + \rho_{\mathrm{h}}) - \rho\ln(\rho) - \rho_{\mathrm{h}}\ln(\rho_{\mathrm{h}})] \tag{2.7}$$

and corresponding expressions hold for all other classes of states. The minimization of the free energy functional must be carried out considering the relations between particle and hole densities derived above, (2.3) or (2.6), and subject to the constraints of constant number of electrons of a given colour. The latter constraints are introduced in a standard way via Lagrange multipliers A_l, by subtracting $\sum_{l=0}^{2S} A_l M^{(l)}$ from the free energy. The corresponding Lagrange multipliers represent the chemical potential ($l = 0$), magnetic field, crystalline field splittings, and so on.

It is useful to introduce an energy potential for each class of excitations, namely

$$\epsilon(k) = T\ln(\rho_{\mathrm{h}}/\rho) \qquad \varphi_n^{(l)}(\Lambda) = T\ln(\sigma_{n\mathrm{h}}^{(l)}/\sigma_n^{(l)}) = T\ln(\eta_{ln}^{(l)}) \tag{2.8}$$

for $l = 1, \ldots, 2S$ and $n = 1, \ldots, \infty$.

We consider ρ and $\sigma_n^{(l)}$ as independent functions and eliminate the 'hole' distributions using (2.3). After some algebra the variation of the free energy yields

$$\begin{aligned}
\epsilon(k) &= k^2 - A_0 - T\sum_{n=1}^{\infty}\int \mathrm{d}\Lambda \frac{1}{\pi}\frac{nc/2}{(\Lambda-k)^2+(nc/2)^2}\ln[1+\eta_n^{(1)}(\Lambda)^{-1}] \\
\ln(1+\eta_n^{(l)}(\Lambda)) &= -\frac{nA_l}{T} + \sum_{m=1}^{\infty}\int \mathrm{d}\Lambda' A_{nm}(\Lambda-\Lambda')\ln[1+\eta_m^{(l)}(\Lambda')^{-1}] \\
&\quad - \sum_{m=1}^{\infty}\int \mathrm{d}\Lambda' C_{nm}(\Lambda-\Lambda')\ln\{[1+\eta_m^{(l+1)}(\Lambda')^{-1}][1+\eta_m^{(l-1)}(\Lambda')^{-1}]\} \\
&\quad l = 1,\ldots,2S \qquad n = 1,\ldots,\infty
\end{aligned} \tag{2.9}$$

where $A_{nm}(\Lambda)$ is the Fourier transform of $\hat{A}_{nm}(\omega)$, $C_{nm}(\Lambda)$ the Fourier transform of $\hat{A}_{nm}(\omega)/[2\cosh(\omega c/2)]$, $\eta_1^{(0)} = \exp(\epsilon/T)$, $\eta_m^{(0)} \equiv \infty$ for $m \geqslant 2$, $\eta_m^{(N)} \equiv \infty$ and the free energy of the Fermi gas is given by

$$F/L = -T\int \frac{\mathrm{d}k}{2\pi}\ln[1+\exp(-\epsilon(k)/T)]. \tag{2.10}$$

Note that the pressure of the electron gas is just $\mathcal{P} = -F/L$.

The density functions can be obtained from the energy potentials ϵ and $\varphi_n^{(l)}$ via

$$\rho(k) = -\frac{1}{2\pi}(1+\mathrm{e}^{\epsilon/T})^{-1}\frac{\partial\epsilon}{\partial\mu} \qquad \sigma_n^{(l)}(\Lambda) = -\frac{1}{2\pi}(1+\eta_n^{(l)})^{-1}\frac{\partial\varphi_n^{(l)}}{\partial\mu} \tag{2.11}$$

where μ is the chemical potential. The Lagrange parameter A_0 plays the role of the chemical potential within the present formulation. The corresponding hole density functions are obtained using (2.8). To prove the relations (2.11) we differentiate (2.9) with respect to A_0; the resulting set of equations is then identical to (2.3) with the use of (2.11). It is also easily verified by using (2.11) that the thermodynamic relation $\partial F / \partial\mu = -N_e$ is satisfied.

Equations (2.9) form an infinite set of recursion relations for the potential functions ϵ and $\varphi_n^{(l)}$. The second set of equations (2.9) is equivalent to the following expressions:

$$\begin{aligned}
\ln[1+\eta_j^{(l)}(\Lambda)^{-1}] &= \sum_{l'=1}^{2S}\int \mathrm{d}\Lambda' G^{\mathrm{c}}_{ll'}(\Lambda-\Lambda')\ln[1+\eta_j^{(l')}(\Lambda')] \\
&\quad - \sum_{l'=1}^{2S}\int \mathrm{d}\Lambda' G_{ll'}(\Lambda-\Lambda')\ln[(1+\eta_{j-1}^{(l')}(\Lambda'))(1+\eta_{j+1}^{(l')}(\Lambda'))] \\
&\quad + \delta_{j,1}\int \mathrm{d}k F_{N-l}(\Lambda-k)\ln[1+\exp(-\epsilon(k)/T)]
\end{aligned} \tag{2.12}$$

where $\eta_0^{(l)} \equiv 0$, and $G_{ll'}$, $G^{\mathrm{c}}_{ll'}$ and F_l are the Fourier transforms of

$$\begin{aligned}
\hat{G}_{ll'}(\omega) &= \frac{\sinh[(N-\max(l,l'))\omega c/2]\sinh[\min(l,l')\omega c/2]}{\sinh(N\omega c/2)\sinh(\omega c/2)} \\
\hat{G}^{\mathrm{c}}_{ll'}(\omega) &= 2\cosh(\omega c/2)\hat{G}_{ll'}(\omega) \qquad \hat{F}_l(\omega) = \frac{\sinh(l\omega c/2)}{\sinh(N\omega c/2)}
\end{aligned} \tag{2.13}$$

respectively. This alternative set of equations (2.12) does not explicitly depend on the Lagrange multipliers A_l; this dependence is then introduced via the asymptotic field boundary conditions

$$\lim_{n\to\infty}\frac{1}{n}\varphi_n^{(l)}(\Lambda) = -A_l. \tag{2.14}$$

As already mentioned earlier there is a formal similarity between the thermodynamic equations for the Fermi gas with a repulsive δ-function interaction and those of the Coqblin–Schrieffer impurity model [17–19] and the SU(N) Heisenberg chain [25]. This similarity is imposed by the common SU(N)-symmetry.

The above set of non-linear integral equations can in general not be solved analytically. In the following section we discuss special limits of these equations.

3. Special cases: repulsive interaction

In this section we discuss several limiting situations of the thermodynamic Bethe *ansatz* equations for a *repulsive* interaction derived in section 2.

3.1. Spin S = 1/2

For $S = 1/2$ we recover the results presented in [7, 8]. Since there is only one spin degree of freedom the superscript l in $\varphi_n^{(l)}$ can be dropped. For $A_0 = A + H/2$ and $A_1 = H$ equations (2.9) reduce to equations (4.10) of [7] if we identify our $\exp(\epsilon(k)/T)$ with $\zeta(k)$ of [7].

3.2. Zero-temperature limit: the ground state

The ground-state integral equations, originally derived by Sutherland [4], can be recovered in the limit $T \to 0$. The Lagrange multipliers A_l for $l = 1, \ldots, 2S$ determine the level splitting between the different colour components. From their definition they are bound to be non-positive (negative or zero). As $T \to 0$ it becomes relevant whether the energy potentials are positive or negative. We follow a procedure similar to that employed in [7] and [12]. We separate the energy potentials into their positive and negative parts, i.e. $\varphi_n^{(l)}(\Lambda) = \varphi_n^{(l)+}(\Lambda) + \varphi_n^{(l)-}(\Lambda)$, $\epsilon(k) = \epsilon^+(k) + \epsilon^-(k)$, such that $\varphi_n^{(l)+}(\Lambda)$, $\epsilon^+(k) > 0$ and $\varphi_n^{(l)-}(\Lambda)$, $\epsilon^-(k) < 0$. Equations (2.9) in the limit $T \to 0$ then yield

$$\begin{aligned}
\epsilon(k) &= k^2 - A_0 + \sum_{n=1}^{\infty}\int \mathrm{d}\Lambda \frac{1}{\pi}\frac{nc/2}{(\Lambda-k)^2+(nc/2)^2}\varphi_n^{(1)-}(\Lambda)\\
\varphi_n^{(l)+}(\Lambda) &= -nA_l - \sum_{m=1}^{\infty}\int \mathrm{d}\Lambda' A_{nm}(\Lambda-\Lambda')\varphi_m^{(l)-}(\Lambda')\\
&\quad + \sum_{m=1}^{\infty}\int \mathrm{d}\Lambda' C_{nm}(\Lambda-\Lambda')[\varphi_m^{(l+1)-}(\Lambda') + \varphi_m^{(l-1)-}(\Lambda')].
\end{aligned} \tag{3.1}$$

Note that $\varphi_m^{(0)} \equiv \delta_{m,1}\epsilon$ and $\varphi_m^{(N)-} \equiv 0$. Using the asymptotic boundary conditions (2.14) it follows that $\varphi_n^{(l)}(\Lambda) > 0$ for all l and $n > 1$ is a possible (and the physical) solution. Hence, all string states are unoccupied and there are no spin bound states in the ground state. All rapidities in (1.4) are then real in the ground state, in agreement with Sutherland's [4] original result. Only $m = n = 1$ terms survive in the sums of (3.1). Below we drop the subindex $n = 1$ in $\varphi_1^{(l)}$, unless explicitly needed.

By inspection we obtain from (3.1) that $\epsilon(k)$ and $\varphi^{(l)}(\Lambda)$ are symmetric functions of their arguments, which monotonically increase with $|k|$ and $|\Lambda|$, respectively. For non-zero density of electrons we must have $\epsilon(k = 0) < 0$. Hence, $\epsilon(k)$ has exactly one zero for $k > 0$, which we denote with Q, i.e. $\epsilon(\pm Q) = 0$; Q increases monotonically with the chemical potential. Note that $\varphi^{(l)}(\pm\infty)$ is either positive or zero, so that these functions have zeros at $\pm B_l$, i.e. $\varphi^{(l)}(\pm B_l) = 0$, where B_l can be ∞. If one $\varphi^{(l)}$ has no zero (i.e. it is always positive) then the system lacks particles of one colour, so that the problem reduces to one with only $N - 1$ degrees of freedom. While Q represents the Fermi surface of the charge rapidities, the B_l play the role of the 'Fermi surface' for the spin waves. In the absence of external fields $A_l = 0$ and $B_l = \infty$ for $l \geqslant 1$, so that the spin-wave bands are completely filled, while $\mu = A_0$.

From the definition of the energy potentials (2.8) it follows that the states for which the potential is negative are occupied and those for which the potential is positive are empty (holes). Differentiating (3.1) with respect to the chemical potential, μ (or A_0), and using (2.11) we obtain

$$\frac{1}{2\pi} = \rho(k) + \rho_{\rm h}(k) - \int_{-B_1}^{B_1} \mathrm{d}\Lambda \frac{1}{\pi} \frac{c/2}{(\Lambda - k)^2 + c^2/4} \sigma^{(1)}(\Lambda)$$

$$\begin{aligned} \sigma^{(l)}(\Lambda) + \sigma_{\rm h}^{(l)}(\Lambda) &+ \int_{-B_l}^{B_l} \mathrm{d}\Lambda' \frac{1}{\pi} \frac{c}{(\Lambda - \Lambda')^2 + c^2} \sigma^{(l)}(\Lambda') \\ &= \int_{-B_{l-1}}^{B_{l-1}} \mathrm{d}\Lambda' \frac{1}{\pi} \frac{c/2}{(\Lambda - \Lambda')^2 + c^2/4} \sigma^{(l-1)}(\Lambda') \\ &\quad + \int_{-B_{l+1}}^{B_{l+1}} \mathrm{d}\Lambda' \frac{1}{\pi} \frac{c/2}{(\Lambda - \Lambda')^2 + c^2/4} \sigma^{(l+1)}(\Lambda') \end{aligned} \tag{3.2}$$

where $\sigma^{(0)} \equiv \rho$ and $\sigma^{(N)} \equiv 0$. These equations are identical to those derived by Sutherland [4]. They are also equivalent to the set (2.3) if thermal equilibrium is imposed on the latter.

The total momentum of the ground state is of course zero. The ground-state energy, the total number of electrons and the number of particles of each colour are obtained through (2.5).

3.3. The limit $c \to \infty$

In the limit of a very large interaction the last line of (2.12) vanishes so that charge and spin degrees of freedom decouple. Consequently the spin waves have no dispersion and the $\eta_n^{(l)}$ are just constants. The remaining Λ' integrations can now straightforwardly be carried out and we obtain from the second set of (2.9)

$$\ln(1 + \eta_n^{(l)}) = -\frac{nA_l}{T} + \sum_{m=1}^{\infty} \min(n, m) \ln \frac{[1 + (\eta_m^{(l)})^{-1}]^2}{[1 + (\eta_m^{(l+1)})^{-1}][1 + (\eta_m^{(l-1)})^{-1}]}. \tag{3.3}$$

Here $\eta_m^{(0)} = \eta_m^{(N)} \equiv \infty$. The solution of this set of algebraic equations depends on the parameters A_l, and can in general not be obtained. For $S = 1/2$ we have $A_1 = -H$, the magnetic field, and an explicit solution has been obtained in [7, 8]. For general spin and pure Zeeman splitting, we have $A_l = -H$ and the algebraic recursion relation can still be solved:

$$1+\eta_n^{(l)} = \frac{\sinh[(n+l)H/2T]\sinh[(n+N-l)H/2T]}{\sinh[lH/2T]\sinh[(N-l)H/2T]}. \tag{3.4}$$

The first equation of (2.9) can also be simplified in the large-c limit. To order $1/c$ we obtain

$$\epsilon(k) = k^2 - A_0 - T\sum_{n=1}^{\infty}\ln[1+(\eta_n^{(1)})^{-1}] = k^2 - \mu - T\ln\frac{\sinh(NH/2T)}{\sinh(H/2T)} \tag{3.5}$$

where we used that $A_0 = \mu + SH$. The last term in (3.5) corresponds just to the free energy of independent spins (dispersionless spin waves) in a magnetic field. The charge dispersion is perfectly parabolic (without mass renormalization), indicating that the charges behave like non-interacting spinless fermions. Indeed, if $c \to \infty$ (hard-core repulsion) Pauli's principle holds as well for electrons with different colours. The free energy is given by (2.10); it corresponds to that of non-interacting fermions with a chemical potential renormalized by the magnetic field.

3.4. The limit $c \to 0$

In the non-interacting limit the Lorentzian integration kernels in (2.9) can be replaced by δ functions, and all the integrations can be carried out immediately. The second set of (2.9) leads to algebraic equations identical to (3.3) except that the charge degrees of freedom do not decouple from the spin waves, i.e. we have to interpret $\eta_m^{(0)}$ as $\delta_{m,1}\exp(\epsilon/T)$. Hence, the spin waves will have a dispersion induced via the charges and the spin wave spectrum is correlated with the Fermi surface of the charges.

For a Zeeman splitting the solution of these algebraic equations is of the same general form as (3.4), except that the argument $nH/2T$ is to be shifted by a quantity λ, i.e. $nH/2T \to nH/2T + \lambda$, where λ contains the dispersion and is determined via

$$\frac{1}{1+\exp(-\epsilon/T)} = \frac{\sinh(H/T+\lambda)\sinh(NH/2T+\lambda)}{\sinh(H/2T)\sinh(SH/T)}. \tag{3.6a}$$

The other relation necessary to determine ϵ and λ self-consistently arises from the first equation of (2.9):

$$\epsilon(k) = k^2 - \mu - T\ln\frac{\sinh(NH/2T+\lambda)}{\sinh(H/2T+\lambda)}. \tag{3.6b}$$

Expressing λ as a function of ϵ from the last equation, one arrives at

$$e^{2\lambda} = e^{-H/T}\,\frac{\exp[(\epsilon-k^2+\mu-SH)/T]-1}{\exp[(\epsilon-k^2+\mu+SH)/T]-1} \tag{3.6c}$$

which inserted into (3.6a) yields the self-consistency equation for $\epsilon(k)$ as a function of k, H, T and μ. For $S = 1/2$ an explicit solution was obtained in [7] and [8].

3.5. High-temperature limit

If the thermal energy is much larger than the interaction strength, the Lorentzian integration kernels in (2.9) can be regarded again as δ functions and the problem reduces to the one discussed in subsection 3.4, except for corrections of order cQ/T.

3.6. *Large-N limit*

In the large-N limit Pauli's principle should become irrelevant, since almost all particles can interact with each other. We are going to consider all colours equally (same number of particles per colour and no external potentials). Hence, if we freeze out and eliminate the spin degrees of freedom, the problem should then essentially reduce to a gas of interacting bosons [1]. Suppressing the internal degrees of freedom means $A_l \equiv 0$ and $\eta_n^{(l)} \to \infty$ for $n > 1$ and $l = 1, \ldots, 2S$.

Using a similar procedure that lead from (2.9) to (2.12) we can arrive at one integral equation for the charges:

$$\epsilon(k) = k^2 - \mu - T \int \mathrm{d}k' F_{2S}^{\exp}(k - k') \ln[1 + \exp(-\epsilon(k')/T)] \tag{3.7a}$$

where $F_{2S}^{\exp}(k)$ is the Fourier transform of $\exp(-|\omega c|/2)\hat{F}_{2S}(\omega)$. In the large-$N$ limit we obtain

$$\epsilon(k) = k^2 - \mu - T \int \mathrm{d}k' \frac{1}{\pi} \frac{c}{(k - k')^2 + c^2} \ln[1 + \exp(-\epsilon(k')/T)] \tag{3.7b}$$

which is indeed the thermodynamic equation for a one-dimensional gas of bosons with a repulsive δ-function interaction [6].

4. Classification of states and thermodynamic equations: attractive interaction

For an *attractive* interaction we proceed in a similar way as in section 2 for the repulsive case. We first classify the solutions of the discrete Bethe *ansatz* equations in the thermodynamic limit according to the string hypothesis [26] and then we derive the thermodynamic Bethe *ansatz* integral equations. Our results are the generalization of those in [7, 8] to arbitrary spin. On the other hand, the integral equations are formally similar to those of the degenerate Anderson impurity model in the limit of infinite Coulomb repulsion [22–24] and the degenerate supersymmetric $t - J$ model in one dimension [27].

4.1. *Classification of states for arbitrary spin*

For an interaction that is *attractive*, $c < 0$, the charge rapidities can now be complex [2, 7, 8], since attractive forces may lead to bound states of electrons of the Cooper-pair type. The solutions of (1.4) are constructed in complete analogy to the Anderson impurity of arbitrary spin in the $U \to \infty$ limit [22–24] and the one-dimensional degenerate supersymmetric $t - J$ model [27]. For large L the states of the system can be classified according to (i) real charge rapidities, belonging to the set $\{\xi_\alpha^{(0)}\}$, associated with unpaired propagating electrons, (ii) complex spin and charge rapidities, which correspond to bound states of electrons with different spin components (colours), and (iii) strings of complex spin rapidities, which represent bound spin states.

Since only electrons with different spin components are scattered, i.e. experience an effective attractive interaction, we may build spin complexes of up to $(2S + 1)$ electrons. A complex of n electrons ($n \leqslant N = 2S + 1$) is characterized by one real $\zeta^{(n-1)}$ rapidity and in general complex $\xi^{(l)}$ rapidities, $l < n - 1$, given by

$$\begin{gathered} \xi_p^{(l)} = \zeta^{(n-1)} + \mathrm{i}pc/2 \qquad l \leqslant n - 1 \leqslant 2S \\ p = -(n - l - 1), -(n - l - 3), \ldots, (n - l - 1). \end{gathered} \tag{4.1}$$

These spin and charge strings form the classes (i) and (ii), which are already present in the ground state [9, 10]. In class (iii) there is a set of strings of complex spin rapidities for each set of rapidities $\{\xi^{(l)}\}$, $l = 1, \ldots, 2S$. A string of length $n-1$ (involving n rapidities) is again given by (2.1).

The above sets of rapidities are inserted into the discrete Bethe *ansatz* equations (1.4) and the resulting coupled equations for the real $\{\zeta^{(l)}\}$ and $\{\Lambda_n^{(l)}\}$ then made logarithmic. The logarithmic equations are defined modulo 2π, yielding sets of integers (half integers) for each class of rapidities. As before these integers play the role of quantum numbers characterizing a state and the quantum numbers within a given set have all to be different giving rise to Fermi–Dirac statistics for all rapidities. In the thermodynamic limit we introduce the usual particle and hole density functions for the rapidities: $\rho^{(l)}(\zeta)$ and $\rho_{\mathrm{h}}^{(l)}(\zeta)$ for the real $\zeta^{(l)}$, $l = 0, \ldots, 2S$, (i.e. the spin-charge bound states) and $\sigma_n^{(l)}(\Lambda)$ and $\sigma_{nh}^{(l)}(\Lambda)$ for the $\Lambda_n^{(l)}$ representing the spin bound states, $l = 1, \ldots, 2S$. In the thermodynamic limit the non-linear equations satisfied by the rapidities are reduced to linearly coupled integral equations for the distribution functions; Fourier transforming the equations we have after some algebra:

$$\begin{aligned}
&\hat\rho_{\mathrm{h}}^{(l)}(\omega)+\hat\rho^{(l)}(\omega)+\sum_{q=0}^{2S}\hat\rho^{(q)}(\omega)\exp[-(l+q-p_{l,q})|\omega c|/2]\{\sinh[(p_{l,q}+1)\omega c/2]/\sinh(\omega c/2)\}\\
&\qquad+\sum_{n=1}^{\infty}\hat\sigma_n^{(l+1)}(\omega)\exp(-n|\omega c|/2)=(l+1)\delta(\omega)\qquad l=0,\ldots,2S\\
&\hat\sigma_{m\mathrm{h}}^{(l)}(\omega)=\hat\rho^{(l-1)}(\omega)\exp(-m|\omega c|/2)+\sum_{n=1}^{\infty}[\hat\sigma_n^{(l-1)}(\omega)+\hat\sigma_n^{(l+1)}(\omega)-2\cosh(\omega c/2)\hat\sigma_n^{(l)}(\omega)]\\
&\qquad\times\exp[-\max(m,n)|\omega c|/2]\{\sinh[\min(m,n)\omega c/2]/\sinh(\omega c/2)\}\\
&\qquad l=1,\ldots,2S\qquad m=1,\ldots,\infty.
\end{aligned}\tag{4.2}$$

The above equations hold with $\sigma_m^{(0)}$, $\sigma_{m\mathrm{h}}^{(0)}$, $\sigma_m^{(N)}$, and $\sigma_{m\mathrm{h}}^{(N)}$ being identically zero, and $p_{l,q} = \min(l,q) - \delta_{l,q}$. The last set of equations (4.2) is equivalent to the following set:

$$\begin{aligned}
&2\cosh(\omega c/2)\hat\sigma_{m\mathrm{h}}^{(l)}(\omega)-\hat\sigma_{m+1\mathrm{h}}^{(l)}(\omega)-\hat\sigma_{m-1\mathrm{h}}^{(l)}(\omega)=\hat\sigma_m^{(l+1)}(\omega)+\hat\sigma_m^{(l-1)}(\omega)-2\cosh(\omega c/2)\hat\sigma_m^{(l)}(\omega)\\
&\qquad m\geqslant 2\\
&2\cosh(\omega c/2)\hat\sigma_{1\mathrm{h}}^{(l)}(\omega)-\hat\sigma_{2\mathrm{h}}^{(l)}(\omega)-\hat\rho^{(l-1)}(\omega)=\hat\sigma_1^{(l+1)}(\omega)\\
&\qquad+\hat\sigma_1^{(l-1)}(\omega)-2\cosh(\omega c/2)\hat\sigma_1^{(l)}(\omega).
\end{aligned}\tag{4.3}$$

These equations are identical to those of the degenerate Anderson impurity model in the $U \to \infty$ limit [22–24] (for a review see [28]) and, except for the driving terms, to the degenerate supersymmetric $t-J$ model in one dimension.

The number of electrons of each spin component, n_l, is given by

$$n_l=\sum_{q=2S-l}^{2S}\int \mathrm{d}\zeta\,\rho^{(q)}(\zeta)+\sum_{n=1}^{\infty}n\int \mathrm{d}\Lambda\left(\sigma_n^{(2S-l)}(\Lambda)-\sigma_n^{(2S+1-l)}(\Lambda)\right)\tag{4.4a}$$

and the total number of electrons becomes

$$N_{\mathrm{e}}/L=\sum_{l=0}^{2S}n_l=\sum_{l=0}^{2S}(l+1)\int \mathrm{d}\zeta\,\rho^{(l)}(\zeta).\tag{4.4b}$$

The energy and momentum of the system are obtained via [10]

$$
\begin{aligned}
E/L &= \sum_{l=0}^{2S}(l+1)\int \mathrm{d}\zeta\left(\zeta^2 - \frac{l(l+2)}{12}c^2\right)\rho^{(l)}(\zeta) \\
P/L &= \sum_{l=0}^{2S}(l+1)\int \mathrm{d}\zeta\,\zeta\rho^{(l)}(\zeta).
\end{aligned}
\tag{4.5}
$$

4.2. *Thermodynamics for arbitrary spin*

The procedure to derive thermodynamic Bethe *ansatz* equations for the *attractive* case is analogous to that employed for the repulsive situation in section 2. We introduce an energy potential for each class of excitations

$$
\begin{aligned}
&\epsilon^{(l)}(\zeta) = T\ln(\rho_{\mathrm{h}}^{(l)}/\rho^{(l)}) \qquad l = 0,\ldots,2S \\
&\varphi_n^{(l)}(\Lambda) = T\ln(\sigma_{n\mathrm{h}}^{(l)}/\sigma_n^{(l)}) = T\ln(\eta_n^{(l)}) \qquad l = 1,\ldots,2S
\end{aligned}
\tag{4.6}
$$

for $n = 1,\ldots,\infty$. In thermal equilibrium these energy potentials are determined by minimizing the free energy functional, $F = E - TS$, with respect to the density functions subject to the constraints (4.4) and the relations (4.2) or (4.3). The energy is given by (4.5), the entropy is determined by the Fermi statistics of the rapidities (see (2.7)) and the constraints (4.4) of constant number of particles of each colour are introduced via Lagrange multipliers A_l, by subtracting $\sum_{l=0}^{2S} A_l n_l$ from the free energy.

There are many equivalent ways to minimize the free energy, depending on which of the density functions are chosen to be independent. If we choose all the $\rho^{(l)}(\zeta)$ and $\sigma_{n\mathrm{h}}^{(l)}(\Lambda)$ as independent functions we arrive at

$$
\begin{aligned}
\epsilon^{(l)}(\zeta) = {} & (l+1)\left(\zeta^2 - \frac{l(l+2)}{12}c^2\right) - \sum_{q=2S-l}^{2S} A_q + T\sum_{q=0}^{2S}\int \mathrm{d}\zeta' \ln\{1+\exp[-\epsilon^{(q)}(\zeta')/T]\} \\
& \times \int \frac{\mathrm{d}\omega}{2\pi}\exp[\mathrm{i}(\zeta-\zeta')\omega - (l+q-p_{l,q})|\omega c|/2] \\
& \times \sinh[(p_{l,q}+1)\omega c/2]/\sinh(\omega c/2) \\
& - T\sum_{n=1}^{\infty}\int \mathrm{d}\Lambda \frac{1}{\pi}\frac{n|c|/2}{(\Lambda-\zeta)^2+(nc/2)^2}\ln\{1+[\eta_n^{(l+1)}(\Lambda)]^{-1}\} \\
& l = 0,\ldots,2S \\
\varphi_n^{(l)}(\Lambda) = {} & T\int \mathrm{d}\Lambda' G_0(\Lambda-\Lambda') \\
& \times \ln\{(1+\eta_{n+1}^{(l)}(\Lambda'))(1+\eta_{n-1}^{(l)}(\Lambda'))/[1+(\eta_n^{(l+1)}(\Lambda'))^{-1}][1+(\eta_n^{(l-1)}(\Lambda'))^{-1}]\} \\
& + T\delta_{n,1}\int \mathrm{d}\zeta\, G_0(\Lambda-\zeta)\ln\{1+\exp[-\epsilon^{(l-1)}(\zeta)/T]\} \\
& l = 1,\ldots,2S \qquad n = 1,\ldots,\infty
\end{aligned}
\tag{4.7}
$$

where $G_0(\Lambda)$ is the Fourier transform of $[\cosh(\omega c/2)]^{-1}$, $\eta_0^{(l)} \equiv 0$, and $\eta_n^{(0)} = \eta_n^{(N)} \equiv \infty$. An equivalent set of equations to the last set of (4.7) is

$$
\ln[1+(\eta_n^{(l)}(\Lambda))^{-1}] = \sum_{q=1}^{2S}\int \mathrm{d}\Lambda' G_{lq}^{\mathrm{c}}(\Lambda-\Lambda')\ln[1+\eta_n^{(q)}(\Lambda')]
$$

$$-\sum_{q=1}^{2S}\int \mathrm{d}\Lambda' G_{lq}(\Lambda-\Lambda')\ln[(1+\eta_{n+1}^{(q)}(\Lambda'))(1+\eta_{n-1}^{(q)}(\Lambda'))]$$
$$+\delta_{n,1}\sum_{q=1}^{2S}\int \mathrm{d}\zeta\, G_{lq}(\Lambda-\zeta)\ln[1+\exp(-\epsilon^{(q-1)}(\zeta)/T)] \tag{4.8}$$

where $G^{\mathrm{c}}_{lq}(\zeta)$ and $G_{lq}(\zeta)$ are defined in (2.13) and $\eta_0^{(q)}\equiv 0$.

If the external (magnetic and crystalline) fields are small, almost all the electrons are going to be in spin-neutral charge bound states of N electrons; it is therefore convenient to distinguish $\epsilon^{(2S)}$ from the other $\epsilon^{(l)}$. The first set of equations of (4.7) can then be rewritten as

$$\begin{aligned}
\epsilon^{(2S)}(\zeta) &= \left(\zeta^2-\mu-\frac{S(S+1)}{3}c^2\right)+T\int \mathrm{d}\zeta'\ln[1+\exp(\epsilon^{(2S)}(\zeta')/T)]\\
&\quad\times\int\frac{d\omega}{2\pi}\exp[\mathrm{i}(\zeta-\zeta')\omega-|\omega c|/2][\sinh(2S\omega c/2)/\sinh(N\omega c/2)]\\
&\quad+T\sum_{q=0}^{2S-1}\int \mathrm{d}\zeta'\ln[1+\exp(-\epsilon^{(q)}(\zeta')/T)]F_{q+1}(\zeta-\zeta')\\
\ln[1+\exp(\epsilon^{(l)}(\zeta)/T)] &= \int \mathrm{d}\zeta'\ln[1+\exp(\epsilon^{(2S)}(\zeta')/T)]F_{l+1}(\zeta-\zeta')\\
&\quad+\sum_{q=0}^{2S-1}\int \mathrm{d}\zeta'\ln[1+\exp(-\epsilon^{(q)}(\zeta')/T)]G^{\mathrm{c}}_{l+1q+1}(\zeta-\zeta')\\
&\quad-\sum_{q=0}^{2S-1}\int \mathrm{d}\Lambda\ln\big(1+\eta_1^{(q+1)}(\Lambda)\big)G_{l+1q+1}(\zeta-\Lambda)
\end{aligned}\tag{4.9}$$

where $F_m(\zeta)$, $G^{\mathrm{c}}_{lq}(\zeta)$ and $G_{lq}(\zeta)$ are defined by (2.13), and the last equation holds for $l=0,\ldots,2S-1$. Here $\mu=\sum_{q=0}^{2S}A_q$ is the chemical potential.

In order to be completely defined, equations (4.8) require asymptotic conditions for the $\varphi_n^{(l)}$ as n tends to infinity, since the equations otherwise are independent of the Lagrange multipliers. These asymptotic boundary conditions are determined by the splitting scheme of the N-fold multiplet through the parameters A_l. We obtain

$$\lim_{n\to\infty}\frac{1}{n}\varphi_n^{(l)}(\Lambda)=A_{2S+1-l}-A_{2S-l}\geqslant 0. \tag{4.10}$$

In particular, for a pure Zeeman splitting it follows from the definition of the magnetization that $A_{2S+1-l}-A_{2S-l}=H$ for $l=1,\ldots,2S$. Note that $A_l-\mu$ is independent of the chemical potential.

The free energy of the system is given by

$$F/L=-T\sum_{l=0}^{2S}(l+1)\int\frac{\mathrm{d}\zeta}{2\pi}\ln[1+\exp(-\epsilon^{(l)}(\zeta)/T)] \tag{4.11}$$

and the pressure of the electron gas is just $\mathcal{P}=-F/L$.

Except for the driving terms the thermodynamic Bethe *ansatz* equations for an attractive interaction are analogous to those of the degenerate Anderson impurity in the $U\to\infty$ limit

[22–24, 28] and the degenerate supersymmetric $t-J$ model [27]. The different driving terms of course modify as well the expressions of the free energy.

The density functions for particles and holes can be derived from the thermodynamic potentials by differentiation with respect to the chemical potential, i.e.

$$\begin{aligned} \rho^{(l)}(\zeta) &= -\frac{1}{2\pi}[1+\exp(\epsilon^{(l)}/T)]^{-1}\frac{\partial\epsilon^{(l)}}{\partial\mu} \\ \sigma_n^{(l)}(\Lambda) &= \frac{1}{2\pi}(1+\eta_n^{(l)})^{-1}\frac{\partial\varphi_n^{(l)}}{\partial\mu} \end{aligned} \tag{4.12}$$

in analogy to (2.11). The corresponding hole-distribution functions are obtained by using (4.6). The correspondence (4.12) can be shown by differentiating (4.7) with respect to μ; using (4.12) they then reduce to the first set of (4.2) and to (4.3). Note that, as expected, differentiating the free energy, (4.11), with respect to the chemical potential, μ, we recover (4.4b), the expression for the total number of particles.

A general solution of the above integral equations cannot be obtained analytically. In section 5 we discuss special limits of these equations.

5. Special cases: attractive interaction

In this section we discuss several limiting situations of the thermodynamic Bethe *ansatz* equations for an *attractive* interaction derived in section 4.

5.1. Spin $S = 1/2$

For $S = 1/2$ there is only one spin degree of freedom and the superscript l in $\varphi_n^{(l)}$ can be dropped. Identifying our $\exp(\epsilon^{(1)}/T)$ with η' in [7] and our $\exp(\epsilon^{(0)}/T)$ with ζ of [7] the set of equations (4.7) reduces to the set (6.10) of [7]. The results presented in [7] and [8] are then contained as a special case.

5.2. *Zero-temperature limit: the ground state*

The ground-state equations for an attractive interaction were originally derived by Takahashi [10]. We now consider the thermodynamic Bethe *ansatz* equations (4.7) in the limit $T \to 0$. As $T \to 0$ it becomes relevant whether the energy potentials are positive or negative and as in section 3 we separate the energy potentials into their positive and negative parts, i.e. $\varphi_n^{(l)}(\Lambda) = \varphi_n^{(l)+}(\Lambda) + \varphi_n^{(l)-}(\Lambda)$, $\epsilon^{(l)}(\zeta) = \epsilon^{(l)+}(\zeta) + \epsilon^{(l)-}(\zeta)$, such that $\varphi_n^{(l)+}(\Lambda)$, $\epsilon^{(l)+}(\zeta) > 0$ and $\varphi_n^{(l)-}(\Lambda)$, $\epsilon^{(l)-}(\zeta) < 0$. Equations (4.7) in the limit $T \to 0$ then yield

$$\begin{aligned} \epsilon^{(l)}(\zeta) &= (l+1)\left(\zeta^2 - \frac{l(l+2)}{12}c^2\right) - \sum_{q=2S-l}^{2S} A_q - \sum_{q=0}^{2S}\int \mathrm{d}\zeta'\epsilon^{(q)-}(\zeta') \\ &\quad\times\int\frac{\mathrm{d}\omega}{2\pi}\exp[\mathrm{i}(\zeta-\zeta')\omega-(l+q-p_{l,q})|\omega c|/2]\sinh((p_{l,q}+1)\omega c/2)/\sinh(\omega c/2) \\ &\quad+\sum_{n=1}^{\infty}\int \mathrm{d}\Lambda\frac{1}{\pi}\frac{n|c|/2}{(\Lambda-\zeta)^2+(nc/2)^2}\varphi_n^{(l+1)-}(\Lambda) \qquad l = 0,\ldots,2S \\ \varphi_n^{(l)}(\Lambda) &= \int \mathrm{d}\Lambda' G_0(\Lambda-\Lambda')[\varphi_{n+1}^{(l)+}(\Lambda') + \varphi_{n-1}^{(l)+}(\Lambda') + \varphi_n^{(l+1)-}(\Lambda') + \varphi_n^{(l-1)-}(\Lambda')] \\ &\quad-\delta_{n,1}\int \mathrm{d}\zeta\, G_0(\Lambda-\zeta)\epsilon^{(l-1)-}(\zeta) \qquad l = 1,\ldots,2S \qquad n = 1,\ldots,\infty. \end{aligned} \tag{5.1}$$

As in (4.7) we have $\varphi_0^{(l)+} = \varphi_m^{(0)-} = \varphi_m^{(N)-} \equiv 0$. Using the asymptotic boundary condition (4.10) it follows that $\varphi_n^{(l)}(\Lambda) > 0$ for all l and $n \geqslant 1$ is a possible (and the physical) solution. Hence, all bands corresponding to spin bound states are empty in the ground state. Only charge and spin-charge bound states survive as $T \to 0$, i.e. the states corresponding to classes (i) and (ii), but not class (iii), in section 4.

By inspection we obtain from (5.1) that all $\epsilon^{(l)}(\zeta)$ are symmetric functions of ζ, which monotonically increase with $|\zeta|$. Hence, these functions must have zeros at $\pm B_l$, i.e. $\epsilon^{(l)}(\pm B_l) = 0$. These zeros correspond to the 'Fermi surfaces' of the various spin-charge bound states. The limits of integration with respect to ζ' in (5.1) are then $-B_q$ and B_q, respectively. Hence, $\epsilon^{(l)-}(\zeta)$ corresponds to $|\zeta| \leqslant B_l$ and $\epsilon^{(l)+}(\zeta)$ is the complementary function.

It is energetically favourable to maximize the number of charge-spin bound states corresponding to strings of length $2S$. These bound states involve N electrons, one of each colour or spin component, and are therefore spin-neutral. In the absence of external fields there is an equal number of particles with each colour, and hence the ground state consists of spin-neutral bound states only. To distinguish the spin-neutral bound states from the spin-dependent ones we denote B_{2S} with Q, which is then determined by the particle density, while all other B_l are equal zero. For small external fields lifting the N-fold degeneracy (e.g. the magnetic field is small compared to the Fermi energy) $\epsilon^{(2S)}(\zeta)$ is determined by the $T \to 0$ limit of the first of equations (4.9). Note that in a zero field $\epsilon^{(l)}(\zeta)$ for $l = 0, \ldots, 2S-1$ is always positive. It requires a finite excitation energy to overcome the energy gap. This energy could be provided by external fields, for example by a Zeeman splitting. Hence, there is a critical magnetic field required to break the spin-neutral charge bound state into smaller bound-state complexes. In the ground state the system does not respond to a field smaller than the lowest critical one.

From the definition of the energy potentials (4.6), it follows that the states for which the potential is negative are occupied and those for which the potential is positive are empty. Differentiating (5.1) with respect to the chemical potential (note that $A_l - \mu$ is independent of μ) and using (4.12) we obtain

$$\rho_{\mathrm{h}}^{(l)}(\zeta) + \rho^{(l)}(\zeta) = \frac{(l+1)}{2\pi} - \sum_{q=0}^{2S} \int_{-B_q}^{B_q} \mathrm{d}\zeta' \rho^{(q)}(\zeta') \times \int \frac{\mathrm{d}\omega}{2\pi} \exp[\mathrm{i}(\zeta - \zeta')\omega - (l + q - p_{l,q})|\omega c|/2] \frac{\sinh[(p_{l,q}+1)\omega c/2]}{\sinh(\omega c/2)}. \tag{5.2}$$

These equations are equivalent to those obtained previously by Takahashi [10] and also correspond to the first set of (4.2) in thermal equilibrium as $T \to 0$.

The number of particles of each spin-component is now given by

$$n_l = \sum_{q=2S-l}^{2S} \int_{-B_q}^{B_q} \mathrm{d}\zeta \rho^{(q)}(\zeta) \tag{5.3}$$

while the total number of electrons and the energy are obtained via (4.4b) and (4.5), respectively. The momentum of the ground state is, of course, zero.

5.3. The limit $c \to -\infty$

In the strong-coupling limit the spin-neutral charge bound states of N electrons become very stable and the critical fields required to split these bound states into smaller complexes are very large. At finite T the thermal energy is then not sufficient to excite the spin-charge bound states of less than N electrons.

Mathematically this is seen from (4.7) as follows. The $G_0(\Lambda)$ and the Lorentzian integration kernels yield contributions of the order of $1/|c|$, so that the $\varphi_n^{(l)}(\Lambda)$ potentials are irrelevant. In the same way, the integration kernel coupling the $\epsilon^{(l)}$ energy potentials leads to contributions of order $1/|c|$, so that the solution of the integral equations is

$$\epsilon^{(l)}(\zeta) = (l+1)\left(\zeta^2 - \frac{l(l+2)}{12}c^2 - \mu\right) \tag{5.4a}$$

where we have neglected external fields. Since $\mu \sim Q^2 - c^2 S(S+1)/3$, only $\epsilon^{(2S)}$ can become negative and only that band can contain electrons. To order $|c|^{-1}$ we obtain for $\epsilon^{(2S)}$

$$\epsilon^{(2S)}(\zeta) = N(\zeta^2 - Q^2) \tag{5.4b}$$

and $\rho^{(2S)}(\zeta) + \rho_{\mathrm{h}}^{(2S)}(\zeta) = N/(2\pi)$, corresponding to free fermions with the dispersion (5.4b) and a density of states enhanced by a factor N.

5.4. The limit $c \to 0$

In the non-interacting limit all integration kernels in (4.7) can be replaced by δ-functions and all integrations can be carried out straightforwardly:

$$\begin{aligned}\epsilon^{(l)}(\zeta) = {} & (l+1)\zeta^2 - \sum_{q=2S-l}^{2S} A_q + T\sum_{q=0}^{2S}(p_{l,q}+1)\ln[1+\exp(-\epsilon^{(q)}(\zeta)/T)] \\ & - T\sum_{n=1}^{\infty}\ln(1+(\eta_n^{(l+1)}(\zeta))^{-1}) \qquad l = 0,\ldots,2S \\ \ln(\eta_n^{(l)}(\Lambda))^2 = {} & \ln\{(1+\eta_{n+1}^{(l)}(\Lambda))(1+\eta_{n-1}^{(l)}(\Lambda))/[1+(\eta_n^{(l+1)}(\Lambda))^{-1}][1+(\eta_n^{(l-1)}(\Lambda))^{-1}]\} \\ & + \delta_{n,1}\ln[1+\exp(-\epsilon^{(l-1)}(\Lambda))] \quad l = 1,\ldots,2S \quad n = 1,\ldots,\infty\end{aligned} \tag{5.5}$$

with $\eta_0^{(l)} \equiv 0$ and $\eta_n^{(0)} = \eta_n^{(N)} \equiv \infty$. An analytic solution of this set of algebraic equations can in general not be obtained. For a pure Zeeman splitting there exists a solution for the second set of (5.5), which is again of the form (3.4) with the replacement $nH/2T \to nH/2T + \lambda$. The parameter λ again contains the dispersion introduced via the potentials $\epsilon^{(l)}$. The second set of (5.5) also requires that $\eta_0^{(l+1)} = \exp(-\epsilon^{(l)}/T)$ for $l = 0,\ldots,2S-1$ in the *ansatz* (3.4). A solution of the form (3.4) has the symmetry $\eta_n^{(l)} = \eta_n^{(N-l)}$, which in turn is not satisfied by the first set (5.5), except for $S = 1/2$. For $S = 1/2$ the solution can be found in [7]. For a larger spin the form (3.4) is not the most general solution of (5.5), but we were unable to obtain a more general analytic expression. As expected, the dispersion of the spin waves is again coupled to the Fermi surface of the charges.

5.5. High-temperature limit

If the thermal energy is very large the variation of the Fermi functions is smooth, so that the integration kernels in (4.7) can again be replaced by δ functions. To order cQ/T the problem then reduces to the one discussed in subsection 5.4.

5.6. Large-N limit

With increasing N, i.e. increasing number of electrons forming the charge bound states, these bound states (for fixed c) become increasingly stable. The bound-state energy increases as N^3 for large N. Their binding energy also grows faster with N as that of the spin-dependent bound states. Hence, the gaps in the excitation spectrum or the critical fields are enhanced with N. The situation is then similar to the $c \to -\infty$ limit. The spin degrees of freedom do not play any relevant role and we obtain

$$\epsilon^{(2S)}(\zeta) = N\left(\zeta^2 - \frac{N^2-1}{12}c^2 - \mu\right) + T\int d\zeta' \frac{1}{\pi}\frac{|c|}{(\zeta-\zeta')^2+c^2}\ln[1+\exp(-\epsilon^{(2S)}(\zeta')/T)]. \tag{5.6}$$

This integral equation is closely related to the one of a Bose gas with δ-function interaction [6], if the chemical potential is redefined as $\mu' = \mu + c^2(N^2-1)/12$ (the only two differences are the factor N multiplying the dispersion and the overall sign of the integral term).

6. Concluding remarks

We have considered the N-fold degenerate gas of electrons with SU(N)-symmetry interacting via a δ-function potential in one dimension. The model is completely integrable and its solution in terms of a sequence of nested Bethe *ansatze* has been constructed by Sutherland [4] (see also [9]). We classified all the states according to the string hypothesis and obtained the thermodynamic Bethe *ansatz* equations for the cases of a repulsive and an attractive interaction. Our results extend previous ones for $S = 1/2$ by Takahashi [7] and Lai [8] to arbitrary spin values. The procedure followed is in close analogy to that used to derive the thermodynamics of the Coqblin–Schrieffer model for a magnetic impurity [17–19] (the repulsive case), and the degenerate Anderson model for a magnetic impurity in the $U \to \infty$ limit [22–24, 28] and the degenerate supersymmetric $t-J$ model in one dimension [27].

We have analysed the thermodynamic Bethe *ansatz* equations in several limiting cases. Our results contain the $S = 1/2$ case [7,8] as a special limit. We further discussed the $T \to 0$, $c \to 0$, $|c| \to \infty$, the high-temperature and large-N limits of the thermodynamic equations. The most relevant properties can be understood from the low-temperature and the large-N limits and are summarized below.

For a *repulsive* interaction there are no bound states in the ground state and the system consists of freely propagating charges (characterized by the real charge-rapidities k) and spin waves (described by the real spin-rapidities). From the $c \to 0$ limit we see that the dispersion of the spin wave excitations is linked to the Fermi surface of the charges. The spin waves become soft in the $c \to \infty$ limit; in this limit the interaction between the fermions becomes a hard-core one and an effective Pauli principle is obtained for all particles, independent of their colour. It is also interesting to notice that in the limit $N \to \infty$ the thermodynamics of the charges reduces to that of a gas of bosons with a repulsive δ-function interaction. The low-T specific heat is proportional to T and the zero-temperature susceptibility is finite [29].

For an *attractive* interaction, on the other hand, the ground state in the absence of symmetry breaking fields consists of spin-neutral charge bound states of N particles, each particle with a different spin-component. Hence, for $N = 2$ these bound states can be interpreted as Cooper pairs, while for $N = 4$ as α particles in a sea of nucleons with the four internal degrees of freedom arising from the direct product of the spin 1/2 and an isospin 1/2. Although these charge bound states are the consequence of a coherent collective state, there is no long-range order in the system at $T = 0$. These charge bound states have a finite binding energy, i.e. a finite external symmetry-breaking potential (for instance a magnetic field in the case of Cooper pairs, reminiscent of the Meissner effect) is required to break the bound state into smaller units (for example, an α particle into two deuterons, or a Cooper pair into two propagating electrons). The excitation spectra of the spin-charge bound states (corresponding to bound states of less than N electrons) have energy gaps, which grow with c^2. The spin-neutral charge bound states exist at all T, in contrast to Cooper pairs in a BCS superconductor, but more like reminiscent of a Bose–Einstein condensation. The $T = 0$ magnetic spin-susceptibility is of course zero (since there is no response to a field smaller than a critical one), but the low-T specific heat is proportional to T.

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